ON THE COMPOSITE.

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1. In the first part of my second paper "On Certain Determinant Forms and their Applications" a general form of the Composite (11) was given, which was

$$\begin{vmatrix} fx & , f_1x & , \ldots, & f_nx & , f_{n+1}x \\ fy_1 & , f_1y_1 & , \ldots, & f_ny_1 & , f_{n+1}y_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fy_p & , f_1y_p & , \ldots, & f_ny_p & , f_{n+1}y_p \\ O'fx_1 & , O'f_1x_1 & , \ldots, & O'f_nx_1 & , O'f_{n+1}x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ O^qfx_q & , O^qf_1x_q & , \ldots, & O^qf_nx_q & , O^qf_{n+1}x_q \\ \Phi(u) & , 0 & , \ldots, & 0 & , 1 \end{vmatrix} = 0, \tag{a}$$

in which O was the symbol of differentiation D, and*

$$\Phi(u) = \left[\frac{d}{dx}\right]_{x=u}^{n=p+q} \begin{vmatrix} fx & , f_1x & , \ldots, & f_nx \\ fy_1 & , f_1y_1 & , \ldots, & f_ny_1 \\ \cdot & \cdot & \cdot & \cdot \\ fy_p & , f_1y_p & , \ldots, & f_ny_p \\ O'fx_1, & O'f_1x_1, & \ldots, & O'f_nx_1 \\ \cdot & \cdot & \cdot & \cdot \\ O^qfx_q, & O^qf_1x_q, & \ldots, & O^qf_nx_q \end{vmatrix} \div \left[\frac{d}{dx}\right]_{x=u}^{n=p+q} \begin{vmatrix} f_1x & , \ldots, & f_{n+1}x \\ f_1y_1 & , \ldots, & f_{n+1}y_1 \\ \cdot & \cdot & \cdot \\ f_1y_p & , \ldots, & f_{n+1}y_p \\ O'f_1x_1, & \ldots, & O'f_{n+1}x_1 \\ \cdot & \cdot & \cdot \\ O^qf_1x_q, & \ldots, & O^qf_{n+1}x_q \end{vmatrix}. \tag{b}$$

2. I propose now to show that the same formula holds good when O is the symbol Δ of Finite Differences.

In my first paper, at the bottom of page 109, I deduced the formula

$$h^nf^n(u) = fx - C_{n,1}f(x-h) + \ldots + c_{n,r}f(x-rh) + \ldots + (-1)^nf(x-nh),$$

$C_{n,r}$ representing the binomial coefficient, and u some value of x lying between x and $x - nh$.

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* In (11), $\left[\frac{d}{dx}\right]_{x=u}^{n=p+q}$ was given as $\left[\frac{d}{dx}\right]_{x=u}^{q+1}$. It is so easy to extend the operation to $n = p + q$ that we do not give it here.

The second member of this equality is the well known interpolation formula of Finite Differences (Boole's Finite Differences, p. 19), and its value is known to be $\Delta^n fx$. We have, therefore,

$$\Delta^n fx = h^n f^n(u)\,. \tag{c}$$

Lagrange's form of Rolle's theorem, written in this notation is

$$\Delta fx = hf'(u)\,.$$

Therefore the above may be regarded as the proper generalization of Lagrange's form, a formula which we now require in our analysis.

Let $O \equiv \Delta$, the symbol of Finite Differences. Let M and N be the minors of 1 and $\Phi(u)$ in (a), respectively. Put

$$M = (-1)^n NR\,,$$

where R is some unknown function of x, and let this equation become

$$M_0 = (-1)^n N_0 R_0\,,$$

wherein we have substituted for the variable x some arbitrary constant x_0. Consider the function

$$F = M - (-1)^n NR_0\,.$$

We have $F = 0$ when $x = x_0, y_1, \ldots, y_p$; therefore the first derivative $F' = 0$ for p values of x, say $u_1, \ldots, u_p$, lying respectively between each pair of values x_0, y_1; y_1, y_2; $\ldots$; y_{p-1}, y_p.

Now $\Delta F = 0$ when $x = x_1$. Hence (if the scale of difference be h), since we have $\Delta Fx = hF'(u)$ (u lying between x and $x + h$), we must have $F' = 0$ for some value of x between x_1 and $x_1 + h$, say $x_1 + h_1$. Therefore F' vanishes $p + 1$ times at the values indicated. It follows, therefore, that F'' vanishes p times, once between each pair of values u_1, u_2; $\ldots$; u_{p-1}, u_p; u_p and $x_1 + h_1$, say when $x = u'_1, \ldots, u'_p$.

Again, since we have

$$\Delta^2 Fx = h^2 F''(u)$$

(u between x and $x + 2h$), and since $\Delta^2 F = 0$ when $x = x_2$, then must $F'' = 0$ for some value of x between x_2 and $x_2 + 2h$, say $x_2 + h_2$. F'' therefore vanishes $p + 1$ times for values of x lying between the pairs of values u'_1, u'_2; $\ldots$; u'_{p-1}, u'_p; u'_p and $x_2 + h_2$.

Reasoning in the same way, we proceed until we show that the qth derivatives of F vanishes $p + 1$ times for values of x which lie between determinate limits; and that, finally, the $(p + q)$th or nth derivative of F vanishes once for some value of x, say u, which lies between the greatest and least of the

quantities

$$x_0, y_1, \ldots, y_p, x_1 + h, \ldots, x_q + qh;$$

so that we have

$$F_{x=u}^{n=p+q} = M_{x=u}^{n=p+q} - (-1)^n N_{x=u}^{n=p+q} R_0 = 0.$$

If we put

$$\Phi(u) = M_{x=u}^{n=p+q} / N_{x=u}^{n=p+q},$$

then

$$M_0 = (-1)^n \Phi(u) N_0;$$

and since x_0 is arbitrary we may drop the subscript and write

$$M = (-1)^n \Phi(u) N,$$

which is formula (a).

It is to be observed that when in (a), $O = D$, the symbol of differentiation, u lies between the greatest and least of $x, x_1, \ldots, x_q, y_1, \ldots, y_p$.

3. The rationale of the process illustrating the application of the composite for differentiation to the expression of functions in an infinite series of other functions may be presented thus :—

Let there be two functions fx and

$$\sum_0^{n-1} A_r\varphi_r x = A_0 + A_1\varphi_1 x + \ldots + A_{n-1}\varphi_{n-1}x.$$

Let the difference between these two functions be R, so that

$$fx = A_0 + A_1\varphi_1 x + \ldots + A_{n-1}\varphi_{n-1}x + R. \tag{c}$$

Let $a_1, \ldots, a_n$ be certain arbitrary values of the variable x, and let us have

$$\left.\begin{array}{l} fa_1 = A_0 + A_1\varphi_1 a_1 + \ldots + A_{n-1}\varphi_{n-1}a_1 + R_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ fa_n = A_0 + A_1\varphi_1 a_n + \ldots + A_{n-1}\varphi_{n-1}a_n + R_n. \end{array}\right\} \tag{d}$$

In these n relations (d) there are n undetermined arbitrary quantities $A_0, A_1, \ldots, A_{n-1}$. Let us determine these so that we shall have $R_1 = 0, \ldots, R_n = 0$. Thus, the value of A_r which satisfies this condition is

$$A = (-1)^r \frac{\begin{vmatrix} fa_1, 1, \varphi_1 a_1, \ldots, \varphi_{r-1}a_1, \varphi_{r+1}a_1, \ldots, \varphi_{n-1}a_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ fa_n, 1, \varphi_1 a_n, \ldots, \varphi_{r-1}a_n, \varphi_{r+1}a_n, \ldots, \varphi_{n-1}a_n \end{vmatrix}}{\begin{vmatrix} 1, \varphi_1 a_1, \ldots, \varphi_{n-1}a_1 \\ \ldots\ldots\ldots \\ 1, \varphi_1 a_n, \ldots, \varphi_{n-1}a_n \end{vmatrix}}.$$

Consider the A coefficients to have these values. Taking now the $n+1$ relations (c) and (d), we have for the value of R,

$$\frac{\begin{vmatrix} fx\,, & 1, & \varphi_1 x\,, & \ldots, & \varphi_{n-1}x \\ fa_1, & 1, & \varphi_1 a_1, & \ldots, & \varphi_{n-1}a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n, & 1, & \varphi_1 a_n, & \ldots, & \varphi_{n-1}a_n \end{vmatrix}}{\begin{vmatrix} 1\,, & \varphi_1 a_1\,, & \ldots, & \varphi_{n-1}a_1 \\ \cdot & \cdot & \cdot & \cdot \\ 1\,, & \varphi_1 a_n, & \ldots, & \varphi_{n-1}a_n \end{vmatrix}} = R\,. \tag{e}$$

We observe that the expansion of the determinant in the numerator of this ratio, according to its first row, gives the coefficient of $\varphi_r x$ the value of A_r as determined above. We observe that this ratio for R takes the indeterminate form $0/0$ when the a's approach a limiting fixed value a. In order to evaluate the limiting value of this ratio as the a's approach the limit a, we apply to the numerator and denominator the operator

$$\left[\frac{d}{da_2}\right]^1_{a_2=a} \cdots \left[\frac{d}{da_n}\right]^{n-1}_{a_n=a},$$

obtaining

$$\frac{\begin{vmatrix} fx & , 1\,, & \varphi_1 x & , \ldots, & \varphi_{n-1}x \\ fa & , 1\,, & \varphi_1 a & , \ldots, & \varphi_{n-1}a \\ f'a & , 0\,, & \varphi'_1 a & , \ldots, & \varphi'_{n-1}a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1}a, & 0\,, & \varphi_1^{n-1}a, & \ldots, & \varphi_{n-1}^{n-1}a \end{vmatrix}}{\begin{vmatrix} \varphi'_1 a & , \ldots, & \varphi'_{n-1}a \\ \cdot & \cdot & \cdot \\ \varphi_1^{n-1}a, & \ldots, & \varphi_{n-1}^{n-1}a \end{vmatrix}} = R\,. \tag{f}$$

It is to be distinctly observed that in this process we do not require the functions fx and $\overset{n-1}{\underset{0}{\Sigma}} A_r\varphi_r x$ to have a contact of the $(n-1)$th order at $x=a$ in order that we may equate their first $n-1$ derivatives when $x=a$. What we require is merely that the functions fx and $\varphi_r x$ $(r=1,\ldots,n-1)$ shall each have a determinate derivative at $x=a$, up to the $(n-1)$th operation. Of course if fx and $\overset{n-1}{\underset{0}{\Sigma}} A_r\varphi_r x$ have an $(n-1)$th contact at a, then our value for R

holds true as well; but it is not dependent on such a relation, it simply includes it.

If now, the successive functions $\varphi_r x$ $(r = 1, \ldots, n)$ may be formed in succession indefinitely according to a given law, so that we may make r in $\varphi_r x$ as great as we choose; then, if it can be shown that R has for its limit 0, as r becomes infinite, and at the same time the A's have limiting values such that $\overset{\infty}{\Sigma} A_r \varphi_r x$ is a convergent series, then we may write

$$fx = a_0 + A_1\varphi_1 x + \ldots \text{ ad. inf.}$$

The value of R has been shown, by the composite, to be

$$\frac{\begin{vmatrix} 1, & \varphi_1 x, & \ldots, & \varphi_n x \\ 1, & \varphi_1 a, & \ldots, & \varphi_n a \\ 0, & \varphi'_1 a, & \ldots, & \varphi'_n a \\ \cdot & \cdot & \cdot & \cdot \\ 0, & \varphi_1^{n-1} a, & \ldots, & \varphi_n^{n-1} a \end{vmatrix}}{\begin{vmatrix} \varphi_1' a, & \ldots, & \varphi_{n-1} a \\ \cdot & \cdot & \cdot \\ \varphi_1^{n-1} a, & \ldots, & \varphi_{n-1}^{n-1} a \end{vmatrix}} \cdot \frac{\left[\dfrac{d}{dx}\right]^n_{x=u} \begin{vmatrix} fx, & 1, & \varphi_1 x, & \ldots, & \varphi_{n-1} x \\ fa, & 1, & \varphi_1 a, & \ldots, & \varphi_{n-1} a \\ f'a, & 0, & \varphi'_1 a, & \ldots, & \varphi'_{n-1} a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1}a, & 0, & \varphi_1^{n-1} a, & \ldots, & \varphi_{n-1}^{n-1} a \end{vmatrix}}{\left[\dfrac{d}{dx}\right]^n_{x=u} \begin{vmatrix} 1, & \varphi_1 x, & \ldots, & \varphi_n x \\ 1, & \varphi_1 a, & \ldots, & \varphi_n a \\ 0, & \varphi'_1 a, & \ldots, & \varphi'_n a \\ \cdot & \cdot & \cdot & \cdot \\ 0, & \varphi_1^{n-1} a, & \ldots, & \varphi_n^{n-1} a \end{vmatrix}} \cdot \quad \text{(g)}$$

The second factor of this is evidently

$$\frac{\begin{vmatrix} f^n u, & \varphi_1^n u, & \ldots, & \varphi_{n-1}^n u \\ f'a, & \varphi_1' a, & \ldots, & \varphi'_{n-1} a \\ \cdot & \cdot & \cdot & \cdot \\ f^{n-1}a, & \varphi_1^{n-1} a, & \ldots, & \varphi_{n-1}^{n-1} a \end{vmatrix}}{\begin{vmatrix} \varphi_1^n u, & \ldots, & \varphi_n^n u \\ \varphi_1' a, & \ldots, & \varphi'_n a \\ \cdot & \cdot & \cdot \\ \varphi_1^{n-1} a, & \ldots, & \varphi_n^{n-1} a \end{vmatrix}}.$$

4. In order to fix these ideas, let us take a concrete example illustrating them. Let it be desired to expand a function in integral powers of the vari-

able. We have by (e)

$$\frac{\begin{vmatrix} fx, & 1, & x, & \dots, & x^{n-1} \\ fa_1, & 1, & a_1, & \dots, & a_1^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n, & 1, & a_n, & \dots, & a_n^{n-1} \end{vmatrix}}{\zeta^{\frac{1}{2}}(a_1 \dots a_n)}, \tag{e$'$}$$

which becomes, after applying the operator

$$\left[\frac{d}{da_2}\right]^1_{a_2=a} \dots \left[\frac{d}{da_n}\right]_{a_n=a},$$

and dividing the numerator and denominator of the result by $(n-1)!!$,

$$\begin{vmatrix} fx & , 1, & \frac{x}{1!}, & \dots, & \frac{x^{n-1}}{(n-1)!} \\ fa & , 1, & \frac{a}{1!}, & \dots, & \frac{a^{n-1}}{(n-1)!} \\ f'a & , 0, & 1, & \dots, & \frac{a^{n-2}}{(n-2)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1}a, & 0, & 0, & \dots, & 1, a \end{vmatrix} = fx - fa - \frac{(x-a)^1}{1!} f'a - \dots - \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}a. \tag{f}$$

The value of this expression is, by the composite through (g), easily seen to be

$$\frac{(x-a)^n}{n!} f^n(u), \qquad (u \text{ between } x \text{ and } a). \tag{g$'$}$$

Thus, we pass from Lagrange's interpolation to Taylor's series. The value of (e$'$) is known to be

$$(x-a_1)\dots(x-a_n)\frac{f^n u}{n!},$$

u lying between the greatest and least of $x, a_1, \dots, a_n$; and when the a's are equal to a, this becomes (g$'$) the expression above given by (g).

The most interesting application of the composite for Finite Differences is to factorial functions or faculties; this we defer for the present.

Another and, in some respects, more satisfactory way of looking at the expression of a function fx in an infinite series of functions $\varphi_1 x, \varphi_2 x, \dots$, may be presented as follows. Consider

$$fx = A_0 + A_1\varphi_1 x + \dots + A_{n-1}\varphi_{n-1}x + R, \tag{I}$$

$$
\left.\begin{array}{l}
f(y+z_0) = A_0 + A_1\varphi_1(y+z_0) + \ldots + A_{n-1}\varphi_{n-1}(y+z_0) + R_0 \\
f(y+2z_1 = A_0 + A_1\varphi_1(y+2z_1) + \ldots + A_{n-1}\varphi_{n-1}(y+2z_1) + R_1 \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
f(y+nz_{n-1}) = A_0 + A_1\varphi_1(y+nz_{n-1}) + \ldots + A_{n-1}\varphi_{n-1}(y+nz_{n-1}) + R_{n-1}
\end{array}\right\} \quad \text{(II)}
$$

In the n relations II we have n undetermined quantities $A_0, \ldots, A_{n-1}$. Let us determine these so that the functions fx and $\Sigma A_r\varphi_r x$ shall coincide for the n values $x = y + z_0, \ldots, y + nz_{n-1}$, or, what is the same thing, so that $R_0 = 0, \ldots, R_{n-1} = 0$.

The value of A_r which satisfies this condition is

$$
(-1)^r \frac{\begin{vmatrix} f(y+z_0), & 1, & \varphi_1(y+z_0), & \ldots, & \varphi_{r-1}(y+z_0), & \varphi_{r+1}(y+z_0), & \ldots, & \varphi_{n-1}(y+z_0) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f(y+nz_{n-1}), & 1, & \varphi_1(y+nz_{n-1}), & \ldots, & \varphi_{r-1}(y+nz_{n-1}), & \varphi_{r+1}(y+nz_{n-1}), & \ldots, & \varphi_{n-1}(y+nz_{n-1}) \end{vmatrix}}{\begin{vmatrix} 1, & \varphi_1(y+z_0), & \ldots, & \varphi_{n-1}(y+z_0) \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \varphi_1(y+nz_{n-1}), & \ldots, & \varphi_{n-1}(y+nz_{n-1}) \end{vmatrix}} . \quad \text{(III)}
$$

Let us consider the A's in I and II to have these values, then

$$
\frac{\begin{vmatrix} fx, & 1, & \varphi_1 x, & \ldots, & \varphi_{n-1}x \\ f(y+z_0), & 1, & \varphi_1(y+z_0), & \ldots, & \varphi_{n-1}(y+z_0) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(y+nz_{n-1}), & 1, & \varphi_1(y+nz_{n-1}), & \ldots, & \varphi_{n-1}(y+nz_{n-1}) \end{vmatrix}}{\begin{vmatrix} 1, & \varphi_1(y+z_0), & \ldots, & \varphi_{n-1}(y+z_0) \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \varphi_1(y+nz_{n-1}), & \ldots, & \varphi_{n-1}(y+nz_{n-1}) \end{vmatrix}} = R. \quad \text{(IV)}
$$

Let $nz = y - w$, y and w being fixed definite values of x. Let $z_0, z_1, \ldots, z_{n-1}$ approach the value z, and therefore the value zero when n approaches infinity. Then when $n = \infty$, the functions fx and $\Sigma A_r\varphi_r x$ coincide in an infinite number of consecutive values of x between y and w. The above value of R, however, in the limit takes the indeterminate form 0/0. For any finite number of rows after the first become identical, the elements in the rth column in these rows being all $\varphi_r y$. Also, any finite number of rows from the bottom are identical, the elements in the rth column in these rows all being $\varphi_r w$. To

remove this indeterminateness, we apply the operator

$$\left[\frac{d}{dz_1}\right]^{1}_{z_1=z}\cdots\left[\frac{d}{dz_{n-1}}\right]^{n-1}_{z_{n-1}=z},$$

to the numerator and denominator and divide by $2^1 \cdot 3^2 \ldots n^{n-1}$. We can now make $z=0$ by making $n=\infty$ without indetermination.

Thus, we have

$$\frac{\left|\begin{array}{llll} fx & ,\ 1\ ,\ \varphi_1 x & ,\ \ldots, & \varphi_{n-1}x \\ f(y+z) & ,\ 1\ ,\ \varphi_1(y+z) & ,\ \ldots, & \varphi_{n-1}(y+z) \\ f'(y+2z) & ,\ 0\ ,\ \varphi_1'(y+2z) & ,\ \ldots, & \varphi'_{n-1}(y+2z) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ f^{r-1}(y+rz) & ,\ 0\ ,\ \varphi_1^{r-1}(y+rz) & ,\ \ldots, & \varphi_{n-1}^{r-1}(y+rz) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ f^{n-r-1}(y+\overline{n-r}.z), & 0\ ,\ \varphi_1^{n-r-1}(y+\overline{n-r}.z), & \ldots, & \varphi_{n-1}^{n-r-1}(y+\overline{n-r}.z) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ f^{n-1}(y+nz) & ,\ 0\ ,\ \varphi_1^{n-1}(y+nz) & ,\ \ldots, & \varphi_{n-1}^{n-1}(y+nz) \end{array}\right|}{\left|\begin{array}{lll} \varphi_1'(y+z) & ,\ \ldots, & \varphi'_{n-1}(y+z) \\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ \varphi_1^{n-1}(y+nz), & \ldots, & \varphi_{n-1}^{n-1}(y+nz) \end{array}\right|} = R. \quad \text{(V)}$$

If now the coefficient of $\varphi_r x$ in the expansion of this determinant be determinate when $n=\infty$, so that the series $\overset{\infty}{\Sigma} A_r\varphi_r x$ is convergent; then, when $n=\infty$, and R is zero when x lies between y and w, the functions fx and $\overset{\infty}{\Sigma} A_r\varphi_r x$ coincide for all values of x between y and w. The composite shows the value of R to be also

$$\frac{\left|\begin{array}{lll} 1\ ,\ \varphi_1 x & ,\ \ldots, & \varphi_n x \\ 1\ ,\ \varphi_1(y+z) & ,\ \ldots, & \varphi_n(y+z) \\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ 0\ ,\ \varphi_1^{n-1}(y+nz), & \ldots, & \varphi_n^{n-1}(y+nz) \end{array}\right| \left|\begin{array}{llll} f^n u & ,\ \varphi_1^n u & ,\ \ldots, & \varphi_{n-1}^{n} u \\ f'(y+2z) & ,\ \varphi_1'(y+2z) & ,\ \ldots, & \varphi_{n-1}(y+2z) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ f^{n-1}(y+nz), & \varphi_1^{n-1}(y+nz), & \ldots, & \varphi_{n-1}^{n-1}(y+nz) \end{array}\right|}{\left|\begin{array}{lll} \varphi'_1(y+z) & ,\ \ldots, & \varphi'_{n-1}(y+z) \\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ \varphi_1^{n-1}(y+nz), & \ldots, & \varphi_{n-1}^{n-1}(y+nz) \end{array}\right| \left|\begin{array}{llll} \varphi^n u & ,\ \varphi_2^n u & ,\ \ldots, & \varphi_n^n u \\ \varphi'_1(y+2z) & ,\ \varphi'_2(y+2z) & ,\ \ldots, & \varphi'_n(y+2z) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ \varphi_1^{n-1}(y+nz), & \varphi_2^{n-1}(y+nz), & \ldots, & \varphi_n^{n-1}(y+nz) \end{array}\right|}, \quad \text{(V)}$$

wherein u lies between y and w, when x lies between these values.

If when $n = \infty$, we have R and z infinitesimals of orders i and j respectively, the functions fx and $\Sigma A_r \varphi_r x$ are said to have coincidence between y and w of the $(i - j + 1)$th order and contact of the $(i - j)$th order. The cases $i \leqq j$ give the so-called "derivativeless" functions, the graph of $\Sigma A_r \varphi_r x$ for $i < j$ being a succession of infinitesimal vertical hachures, whose length is of order i, drawn on the continuous curve fx, and separated by the infinitesimal interval z, of order $j > i$.

If the φ functions be periodic functions having for periods $w - y$ and submultiples of $w - y$, then, in general, $\varphi_r^p y = \varphi_r^p w$.

When we make n infinite, then, we have under these circumstances

$$\frac{\begin{vmatrix} fx, & 1, & \varphi_1 x, & \varphi_2 x, & \cdots \\ fy, & 1, & \varphi_1 y, & \varphi_2 y, & \cdots \\ f'y, & 0, & \varphi'_1 y, & \varphi'_2 y, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^r y, & 0, & \varphi_1^r y, & \varphi_2^r y, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}}{\begin{vmatrix} \varphi'_1 y, & \varphi'_2 y, & \cdots \\ \cdot & \cdot & \cdot \\ \varphi_1^r y, & \varphi_2^r y, & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}} = 0. \qquad \text{(VII)}$$